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A modal linear logic

— its proof theory and semantics —

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Linear logic and its "modal" extension

Linear logic as a refinement of intuitionistic logic

$A_1, \ldots, A_n \vdash C$
in intuitionistic logic

$\xmapsto{(-)^\circ}$

$!(A_1)^\circ, \ldots, !(A_n)^\circ \vdash (C)^\circ$
in linear logic

Modal linear logic as a refinement of intuitionistic modal logic

$\Box A_1, \ldots, \Box A_n, B_1, \ldots, B_m \vdash C$

in modal logic

$\xmapsto{(-)^\circ}$

$(\Box A_1, \ldots, \Box A_n, B_1, \ldots, B_m \vdash C)^\circ$
in **"modal linear logic"**

Today's talk

Introduction

① Linear logic as a refinement of ordinary logic

Previous work
[F. & Yoshimizu '19]

② A "modal linear logic" for intuitionistic S4

a subsystem of *subexponential linear logic*

[Danos et al. '95][Nigam et al. '11]

Recent result

③ A categorical model for the modal linear logic

an adjoint model based on the LNL model [Benton '95]

Linear Logic: a refinement of ordinary logic

Linear logic

Linear logic is ...

- a "resource-sensitive" logic that every assumption is used "linearly" unless it is tagged with the !-modality
  - $!A$ roughly means "infinitely many" (possibly zero) assumptions of A
- a "refinement" of ordinary logic (classical logic / intuitionistic logic) in the sense that the ordinary logics can be embedded into L.L. via a translation

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Judgments with Linear Connectives

- $A, (A \multimap B) \vdash B$
- $A, (A \multimap B) \nvdash A \otimes B$
- $A \nvdash A \otimes A$
- $A, B \nvdash A$

Judgments with Exponentials

- $!A, (A \multimap B) \vdash B$
- $!A, (A \multimap B) \vdash !A \otimes B$
- $!A \vdash !A \otimes !A$
- $A, !B \vdash A$

⚠ This talk only deals with the $(\otimes, \multimap, !)$-fragment in the intuitionistic setting.

A proof theory of linear logic: Intuitionistic MELL

The syntax of **MELL** (*Multiplicative Exponential Linear Logic*) is as follows:

Grammar

$$A, B ::= p \mid A \otimes B \mid A \multimap B \mid !A$$

$$\Gamma, \Delta ::= \{A_1, \ldots, A_n\} \quad (\text{ a multi-set of formulae })$$

Judgment

$$\Gamma \vdash A$$

- To prove the judgment $B_1, \ldots, B_n \vdash A$, we basically need to consume each of the formulae $B_1, \ldots, B_n$ exactly once.

Inference rule of MELL

Axiom / Cut

$$\frac{}{A \vdash A}\text{Ax} \qquad \frac{\Gamma \vdash A \qquad A, \Delta \vdash B}{\Gamma, \Delta \vdash B}\text{Cut}$$

Linear connectives

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\otimes L \qquad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\otimes R \qquad \frac{\Gamma \vdash A \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}\multimap L \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\multimap R$$

Exponential modality

$$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\text{W} \qquad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\text{C} \qquad \frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\text{D} \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\text{P}$$

$$(\, !\Gamma \overset{\text{def}}{=} \{!A \mid A \in \Gamma\} \,)$$

Examples of derivation in linear logic

Derivable judgment: $A, (A \multimap B) \vdash B$

$$\dfrac{\dfrac{}{A \vdash A}\,\text{Ax} \qquad \dfrac{}{B \vdash B}\,\text{Ax}}{A, (A \multimap B) \vdash B}\multimap L$$

Underivable judgment: $A, (A \multimap B) \nvdash A \otimes B$

$$\dfrac{\dfrac{}{A \vdash A}\,\text{Ax} \qquad \dfrac{\vdots}{B \vdash A \otimes B}}{A, (A \multimap B) \vdash A \otimes B}\multimap L \qquad \dfrac{\dfrac{}{A \vdash A}\,\text{Ax} \qquad \dfrac{\vdots}{A \multimap B \vdash B}}{A, (A \multimap B) \vdash A \otimes B}\otimes R$$

Examples with the exponential

Derivable judgment: $!A, (A \multimap B) \vdash !A \otimes B$

$$\dfrac{\dfrac{\dfrac{}{!A \vdash !A}\text{Ax} \qquad \dfrac{\dfrac{\dfrac{}{A \vdash A}\text{Ax}}{!A \vdash A}\text{D} \qquad \dfrac{}{B \vdash B}\text{Ax}}{!A, (A \multimap B) \vdash B}\multimap L}{!A, !A, (A \multimap B) \vdash !A \otimes B}\otimes R}{!A, (A \multimap B) \vdash !A \otimes B}\text{C}$$

Property on MELL

Theorem. Cut-elimination(cf. [Lincoln et al. 1992])

If a judgment $\Gamma \vdash A$ is derivable, then there exists a *cut-free* proof for the same judgment.

Theorem. Embedding from intuitionistic logic

If a judgment $\Gamma \vdash A$ of LJ (intuitionistic prop. logic) is derivable, then there exists the corresponding proof in MELL.

Girard translation

Intuitionistic Prop. Logic

$$\begin{array}{c} \vdots\, \mathcal{D} \\ \Gamma \vdash A \end{array}$$

in LJ

$$\xmapsto{(-)^\circ}$$

Intuitionistic MELL

$$\begin{array}{c} \vdots\, (\mathcal{D})^\circ \\ (\Gamma \vdash A)^\circ \end{array}$$

in MELL

Definition. the (call-by-name) Girard translation [Girard '87][Maraist et al. '99]

Formula

$$(p)^\circ \stackrel{\text{def}}{=} p$$

$$(A \wedge B)^\circ \stackrel{\text{def}}{=} A^\circ \otimes B^\circ$$

$$(A \to B)^\circ \stackrel{\text{def}}{=} !(A^\circ) \multimap (B^\circ)$$

Judgment

$$(A_1, \ldots, A_n \vdash B)^\circ \stackrel{\text{def}}{=} !A_1{}^\circ, \ldots, !A_n{}^\circ \vdash B^\circ$$

Modal Linear Logic

Towards "modal linear logic"

From LJ to MELL: Linear Logic as a refinement of Intuitionistic Logic

$$\begin{array}{c}\vdots\ \mathcal{D}\\ \Gamma \vdash A\end{array} \text{ in LJ} \quad \overset{(-)^\circ}{\longmapsto} \quad \begin{array}{c}\vdots\ (\mathcal{D})^\circ\\ (\Gamma \vdash A)^\circ\end{array} \text{ in MELL}$$

Its modal extension: *"Modal Linear Logic"* as a refinement of Modal Logic

$$\begin{array}{c}\vdots\ \mathcal{D}\\ \Box\Delta, \Gamma \vdash A\end{array} \text{ in } \mathrm{LJ}^{\Box} \quad \overset{(-)^\circ}{\longmapsto} \quad \begin{array}{c}\vdots\ (\mathcal{D})^\circ\\ (\Box\Delta, \Gamma \vdash A)^\circ\end{array}$$

in *"modal linear logic"*

[Troelstra & Schwichtenberg '96]

On the naïve attempt for "modal linear logic"

If we introduce "modal linear logic" as a linear logic with a $\Box$-modality:

$$A ::= p \mid A \multimap B \mid !A \mid \Box A$$

$$\Delta, \Gamma, \Sigma ::= \{A_1, \ldots, A_n\} \quad (\text{ a multi-set of formulas })$$

with the $\Box$-rules (e.g., as in [Troelstra & Schwichtenberg '96])

$$\frac{\Gamma, A \vdash B}{\Gamma, \Box A \vdash B} \quad \frac{\Box\Gamma \vdash A}{\Box\Gamma \vdash \Box A}$$

where $\Box\Gamma \stackrel{\text{def}}{=} \{\Box A \mid A \in \Gamma\}$

$$\frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B} \quad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}$$

Fact: We cannot give a Girard trans. from modal logic to this logic

Problem on the naïve formulation

The problem stems from the "non-canonicity" on the modalities

(cf. [Schellinx '94][Baelde '08]; and many studies on "proof-theoretic semantics")

Canonicity on the logical connectives

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\otimes L \qquad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\otimes R$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \bigstar B \vdash C}\bigstar L \qquad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \bigstar B}\bigstar R$$

Then, $A \otimes B \vdash A \bigstar B$ and $A \bigstar B \vdash A \otimes B$ are derivable.

Non-canonicity on the modalities

$$\frac{\Gamma, A \vdash B}{\Gamma, \Box A \vdash B} \qquad \frac{\Box\Gamma \vdash A}{\Box\Gamma \vdash \Box A}$$

$$\frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B} \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}$$

Then, $\Box A \vdash !A$ and $!A \vdash \Box A$ are not derivable.

Solution to the problem

From the observation of *subexponential linear logic* [Danos et al. '95][Nigam et al. '11]
and *adjoint logic* [Reed '09][Licata et al. 16][Pruiksma et al. '18],

Modalities (i.e., exponentials) must be layered w.r.t. a preorder $\preceq$

General inference rule

For multi-exponentials (called *subexponentials*) $\{!_i\}_{i \in I}$ with $!_0 \prec !_1 \prec \cdots \prec !_n$, we define the promotion rule by

$$\frac{!_n X, \ldots, !_{(k+1)} Y, !_k Z \vdash A}{!_n X, \ldots, !_{(k+1)} Y, !_k Z \vdash !_k A}\ \text{Prom. for } !_k$$

Sequent calc. of the modal linear logic S4 [F. & Yoshimizu '19]

The **modal linear logic**, called $\mathrm{MELL}^{\boxed{!}}$, is introduced as an intuitionistic fragment of subexponential L.L. with two subexponentials ($!, \boxed{!}$)

Syntax

$$A ::= p \mid A \otimes B \mid A \multimap B \mid !A \mid \boxed{!}A$$

$$\Delta, \Gamma ::= \{A_1, \ldots, A_n\} \quad \text{(a multi-set of formulae)}$$

Inference rule

$$\frac{\Gamma, A \vdash B}{\Gamma, \boxed{!}A \vdash B} \qquad \frac{\boxed{!}\Delta \vdash A}{\boxed{!}\Delta \vdash \boxed{!}A} \qquad \frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B} \qquad \frac{\boxed{!}\Delta, !\Gamma \vdash A}{\boxed{!}\Delta, !\Gamma \vdash !A}$$

(with the weakening and contraction rules for ! and $\boxed{!}$)

$$\frac{}{A \vdash A} \qquad \frac{\Gamma \vdash A \quad A, \Gamma' \vdash B}{\Gamma, \Gamma' \vdash B} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C} \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \qquad \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}$$

Property: Cut-elimination theorem

Theorem. Cut-elimination

If a judgment $\Gamma \vdash A$ is derivable in $\mathrm{MELL}^{\boxed{!}}$, then there is a *cut-free* proof for the same judgment.

Proof. (sketch)

By simultaneous induction, we show that the followings are admissible:

$$\frac{\Gamma \vdash A \qquad A, \Delta \vdash B}{\Gamma, \Delta \vdash B} \qquad \frac{\Gamma \vdash !A \qquad (!A)^n, \Delta \vdash B}{\Gamma, \Delta \vdash B} \qquad \frac{\Gamma \vdash \boxed{!}A \qquad (\boxed{!}A)^n, \Delta \vdash B}{\Gamma, \Delta \vdash B}$$

where $(C)^n$ means the multi-set that has n-occurences of C.

Property: Modal Girard translation

Theorem. Embedding from modal logic

If a judgment $\Gamma \vdash A$ is derivable in $\mathrm{LJ}^{\Box}$, then there is a coressponding proof in $\mathrm{MELL}^{\boxed{!}}$.

Proof. By modal Girard translation.

By mapping a derivation $\Box\Delta, \Gamma \vdash A$ to $\boxed{!}(\Delta)^\circ, !(\Gamma)^\circ \vdash A^\circ$, where

$$
\begin{aligned}
(p)^\circ &\overset{\text{def}}{=} p \\
(A \land B)^\circ &\overset{\text{def}}{=} A^\circ \otimes B^\circ \\
(A \to B)^\circ &\overset{\text{def}}{=} !(A^\circ) \multimap (B^\circ) \\
(\Box A)^\circ &\overset{\text{def}}{=} \boxed{!}(A^\circ)
\end{aligned}
$$

A semantics for the modal linear logic

On the semantics

Derivation in the modal linear logic

$$\begin{array}{c} \vdots\,\mathcal{D} \\ \Gamma \vdash A \end{array}$$

in $\mathrm{MELL}^{\boxed{!}}$

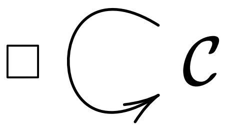
Denotation of the derivation

$$\left[\!\left[\begin{array}{c} \vdots\,\mathcal{D} \\ \Gamma \vdash A \end{array} \right]\!\right]$$

in "some structure"

Models of modal logic and linear logic

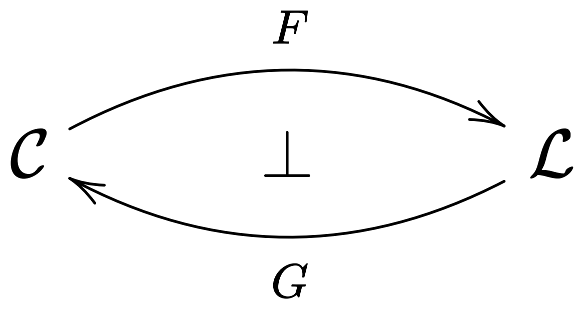
Modal category for S4 (e.g., [Hofmann '99][de Paiva & Ritter '16][Kavvos '20])



- $\mathcal{C}$: cartesian closed category
- $\Box$: an endo-functor on $\mathcal{C}$

  (a monoidal comonad)

Linear-Non-Linear (LNL) model [Benton '95]



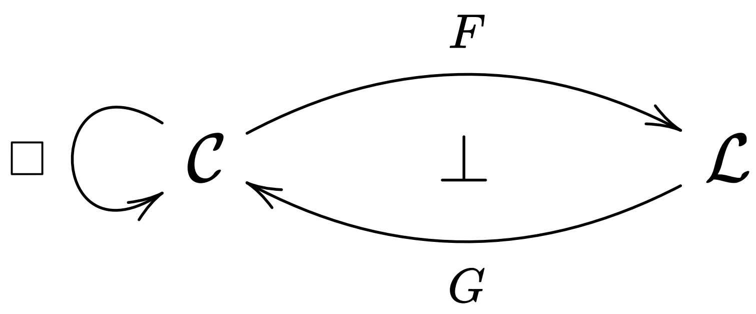
- $\mathcal{C}$: cartesian closed category
- $\mathcal{L}$: symmetric monoidal closed category

  where (F, G) forms a sym. mon. adjunction
- $! \stackrel{\text{def}}{=} F \circ G$

  (*Linear Exponential Comonad*)

Adjoint model of the modal linear logic

Model of the modal linear logic



- $\mathcal{C}$: CCC
  - $\Box$ is a product-preserving comonad
- $\mathcal{L}$: SMCC
- (F, G) forms a symmetric monoidal adjunction

Interpretation (in $\mathcal{L}$)

- For formulae:
  - $[\![!A]\!] \overset{\text{def}}{=} FG[\![A]\!]$
  - $[\![\boxed{!}A]\!] \overset{\text{def}}{=} F\Box G[\![A]\!]$
  - etc
- For derivaitons:
  $[\![\boxed{!}\Delta, !\Gamma, \Sigma \vdash D]\!]$ is given by
  $(\bigotimes_{A_i \in \Delta} F\Box G[\![A_i]\!])$
  $\otimes(\bigotimes_{B_j \in \Gamma} FG[\![B_j]\!])$
  $\otimes(\bigotimes_{C_k \in \Sigma} [\![C_k]\!]) \longrightarrow [\![D]\!]$

Concluding remark

Research related to "modal linear logic"

"A modal view of linear logic" [Martini & Masini '94]

- A translation from classical S4 into full (i.e., multiplicative-additive-exponential) linear logic, using the Grisin–Ono translation

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"A linear approach to modal proof theory" [Schellinx '96]

- A "proof-normalization"-preserving translation from classical S4 into "bi-colored linear logic" with subexponential pairs $\langle !_0, ?_0 \rangle$ and $\langle !_1, ?_1 \rangle$
- The translation is given as an extension of "linear decoration"

[Danos–Joitnet–Schellinx '93]

"On the cut-elimination of the modal μ-calculus: Linear Logic to the rescue"

[Bauer & Saurin '25]

- A "modal linear" μ-calculus to discuss the cut-elimination theorem for the modal μ-calculus

Summary

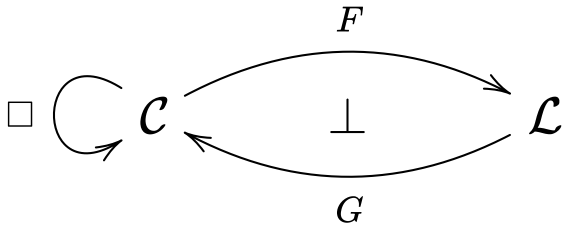
Modal linear logic is an integration of (S4) modal logic and linear logic

Proof theory (λ-calc. under Curry–Howard)

- sequent calculus: MELL + $\boxed{!}$-modality
- modal linear λ-calc. (not mentioned in this talk)

$\lambda^{\square}$ [Pfenning & Davies '01] + DILL [Barber & Plotkin '97]

Model



- $\mathcal{C}$: CCC with a monoidal comonad $\square$
- $\mathcal{L}$: SMCC
- (F, G) : a sym. mon. adjunction
- $! \overset{\text{def}}{=} FG; \quad \boxed{!} \overset{\text{def}}{=} F\square G$

Some interesting directions (?)

- Extension to multi-modalities
  ("adjoint logic" / "subexponential linear logic")
- Extension to graded modalities, bounded exponentials, etc.
  ("graded modal logic" / "bounded linear logic")

Today's slide

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Appendix

Reference

[Baelde 2008] A Linear Approach to the Proof-theory of Least and Greatest Fixed Points
[Barber 1996] Dual Intuitionistic Linear Logic
[Barber & Plotkin 1997] Dual Intuitionistic Linear Logic (unpublished note)
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[Girard 1987] Linear logic
[Hofmann 1999] Semantics of linear/modal lambda calculus
[Kavvos 2020] Dual-Context Calculi for Modal Logic
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[Lincoln et al. 1992] Decision problems for propositional linear logic
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[Martini & Masini 1994] A Modal View of Linear Logic
[Nigam et al. 2011] Specifying Proof Systems in Linear Logic with Subexponentials
[Pfenning & Davies 2001] A judgmental reconstruction of modal logic
[Pruiksma et al. 2018] Adjoint Logic
[Reed 2009] A Judgmental Deconstruction of Modal Logic
[Schellinx 1994] The Noble Art of Linear Decorating
[Schllinx 1996] A Linear Approach to Modal Proof Theory
[Troelstra & Schwichtenberg 1996] Basic Proof Theory

Typed λ-calc. for the modal linear logic [F. & Yoshimizu '19]

A typed λ-calc. for modal linear logic, an instance of the adjoint model

- It is defined as an integration of $\lambda^{\square}$ [Pfenning & Davies '00] / DILL [Barber '96]
- It corresponds to a natural deduction for modal linear logic under the Curry–Howard correspondence

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Syntax

$$A ::= p \mid A \multimap B \mid !A \mid \boxed{!}A$$

$$M ::= x \mid \lambda x : A.\, M \mid M\, N$$

$$\mid\ !M \mid \mathbf{let}\ !x \Leftarrow M\ \mathbf{in}\ N$$

$$\mid \boxed{!}M \mid \mathbf{let}\ \boxed{!}x \Leftarrow M\ \mathbf{in}\ N$$

$$\Delta, \Gamma, \Sigma ::= \{x_1 : A_1, \ldots, x_n : A_n\}$$

Typing Judgment

$$\Delta; \Gamma; \Sigma \vdash M : A \qquad (\approx\ \boxed{!}\Delta, !\Gamma, \Sigma \vdash A\ )$$

Typing rules of the modal linear λ-calc.

Rules for $\multimap$

$$\frac{\Delta; \Gamma; \Sigma, x : A \vdash M : B}{\Delta; \Gamma; \Sigma \vdash \lambda x.\, M : A \multimap B} \multimap\text{-I}$$

$$\frac{\Delta; \Gamma; \Sigma \vdash M : A \multimap B \qquad \Delta; \Gamma; \Sigma \vdash N : A}{\Delta; \Gamma; \Sigma, \Sigma' \vdash MN : B} \multimap\text{-E}$$

Rules for ! and $\boxed{!}$

$$\frac{\Delta; \Gamma; \emptyset \vdash M : A}{\Delta; \Gamma; \emptyset \vdash !M : !A} !\text{-I} \qquad\qquad \frac{\Delta; \emptyset; \emptyset \vdash M : A}{\Delta; \emptyset; \emptyset \vdash \boxed{!}M : \boxed{!}A} \boxed{!}\text{-I}$$

$$\frac{\Delta; \Gamma; \Sigma \vdash M : \boxed{!}B \qquad x : A, \Delta; \Gamma; \Sigma' \vdash N : B}{\Delta; \Gamma; \Sigma, \Sigma' \vdash \mathbf{let}\ \boxed{!}x \Leftarrow M\ \mathbf{in}\ N : B} \boxed{!}\text{-E}$$

(The rule !-E is defined simiarly to $\boxed{!}$-E)

Reduction

Reduction rule

$$
\begin{aligned}
(\overline{\lambda} x : A.\, M)\ N &\rightsquigarrow M[N/x] \\
(\mathbf{let}\ !x \Leftarrow !N\ \mathbf{in}\ M) &\rightsquigarrow M[N/x] \\
(\mathbf{let}\ \boxed{!}x \Leftarrow \boxed{!}N\ \mathbf{in}\ M) &\rightsquigarrow M[N/x]
\end{aligned}
$$